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THE COLLINEATIONS OF SPACE WHICH TRANSFORM A NON-DEGENERATE QUADRIC SURFACE INTO ITSELF.*

BY RUTH G. WOOD.

THE object of this paper is to discuss the ∞^6 collineations of space which transform a non-degenerate quadric surface into itself. These fall into two classes. The collineations of the *first* kind leave the two systems of generators of the quadric invariant, while the collineations of the *second* kind interchange the two systems. When the quadric surface degenerates into the imaginary circle at infinity, these collineations of the first and second kinds become respectively the displacements and the symmetry transformations of euclidean geometry. We shall then call a collineation of the first kind a non-euclidean displacement, and a collineation of the second kind a non-euclidean symmetry transformation. These transformations for euclidean geometry have been discussed by Mr. Gale in a paper entitled† *Wiener's Theory of Displacements, with Application to the Proof of Four Theorems of Chasles.* The methods here employed are similar to those used by Mr. Gale, while the theorems obtained correspond in non-euclidean space to the four theorems of Chasles proved in his paper.

This correspondence is readily seen if we choose the quadric surface for the absolute. Two lines are then said to be conjugate polars with respect to the absolute when the polar planes of all the points of one line pass through the other. Two planes are said to be conjugate or perpendicular when the pole of one lies in the other. Two intersecting lines are said to be perpendicular when each intersects the conjugate of the other, and a line is said to be perpendicular to a plane when it passes through the pole of the plane.

The fundamental operation in this discussion is the involutory transformation known as a skew reflection on conjugate polars of the quadric. By this operation a point P is transformed into a point P' such that the line PP' is divided harmonically by the two polars, which are called the *directrices* of the skew reflection. When the quadric degenerates into the imaginary circle at infinity the skew reflection becomes reflection in a line, the transformation used by Mr. Gale in his discussion. We shall then call a skew reflection on conjugate polars of the quadric a *non-euclidean line reflection*.

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* This paper was read before the American Mathematical Society, December 28, 1900.

† ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1900), p. 1.

We shall first show that any non-euclidean displacement may be resolved into the product of two non-euclidean line reflections. This affords a simple method of resolving non-euclidean displacements and symmetry transformations so that some of their fundamental properties are at once evident. From these resolutions theorems concerning the displacements of straight lines and plane figures in non-euclidean space are readily deduced, which, as before mentioned, become well known theorems of Chasles when the quadric degenerates into the imaginary circle at infinity.

NON-EUCLIDEAN DISPLACEMENTS.

Non-euclidean displacements leave the two systems of generators of a quadric surface invariant and may be classified according to the invariant figure upon the invariant quadric. The resolution into skew reflections depends essentially upon a theorem of Wiener concerning projectivities.

A projectivity upon a line is the projective relation between the points A, B, C . . . and the points A', B', C' . . . of that line, and may be denoted by the symbol $ABC\pi A'B'C'$. The two points on the line which correspond to themselves are called double points. If P, Q are corresponding points, and D, D' the double points of a projectivity, the cross ratio (DD', PQ) is constant. When this cross ratio is harmonic the projectivity is said to be an involution. Wiener's theorem above referred to is:

Any projectivity $DD'P\pi DD'Q$ *may be resolved in* ∞^1 *ways into the product of two involutions.*[*]

Proof. Any point upon PQ may be chosen as a double point D_1 of the first involution. The second double point D_1' is then determined as the fourth harmonic of D, D' and D_1. In this involution the point P will correspond to a point P'. The double points D_2, D_2' of the second involution are now chosen to divide harmonically D, D' and P', Q.

If D and D' fall together in D, so that $D^2P\pi D^2Q$, then D and any point D_1 upon PQ are chosen as the double points of the first involution. In this involution the point P will correspond to a point P'. The double points of the second involution are D_2 and D, D_2 being chosen as the fourth harmonic of $D\,P'$ and Q.

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[*] *Leipziger Berichte*, vol. 43 (1891), p. 648.

This being established we may now state our

FUNDAMENTAL THEOREM: *Any non-euclidean displacement may be resolved into the product of two non-euclidean line reflections.*

We will give the proof for the following five types of displacements, which are evidently all that can occur.

(*a*) *Two generators of each system remain fixed.** The conjugate polars which are the opposite edges of the tangential tetrahedron formed by the four fixed generators we shall call the axes of this displacement. The displacement depends upon fourteen parameters and is completely determined by the invariant quadric S, the vertices A, B, C, D of the tangential tetrahedron and a pair of corresponding general points $P P'$ which lie upon S. Through P and P' draw the lines intersecting the axes AD and BC in L, L' and M, M' respectively. Then upon AD and BC are established projectivities such that $ADL \pi ADL'$ and $BCM \pi BCM'$. Each of these projectivities may be resolved in ∞^1 ways into the product of two involutions. The lines joining in pairs the double points of these involutions are conjugate polars with respect to the quadric and may be taken as the directrices of two non-euclidean line reflections, the product of which is a displacement which leaves the quadric S invariant. Since the double points may be joined in four ways, four different displacements are thus obtained each of which leaves the tetrahedron $ABCD$ invariant and transforms one intersection of the line LM with the quadric into an intersection of the line $L'M'$ with the quadric. Hence one of these displacements transforms the point P into the point P' and is consequently the given displacement. It may be found by at most four trials. The resolution of this displacement into the product of two line reflections may thus be effected in ∞^2 ways.

(*b*) *Two generators of one system and one of the other remain fixed.*

This displacement has two fixed points, the intersections of the fixed generators. It depends upon thirteen parameters and is completely defined by the quadric S, the two fixed points A, B and a pair of corresponding points P, P' on S. Through P and P' pass the generators which intersect the fixed generator AB in the points L and L'. Then upon AB is established a projectivity $ABL \pi ABL'$ which may be resolved in ∞^1 ways into the product of two involutions with double points D_1, D_1' and D_2, D_2'. Through D_1 and

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* The fundamental theorem for case (*a*) has been stated by Mr. Wilson, *Trans. Amer. Math. Soc.*, vol. 1 (1900), p. 196

D_1' draw any pair of conjugate polars, which may be done in ∞^1 ways. By a reflection on these lines as directrices, the point P is transformed into a point P'' on the quadric. The directrices of the second line reflection are conjugate polars through D_2 and D_2' and must be chosen so as to transform P'' into P'.

This may be done as follows: Through D_2 draw the generator of the first system and note the points M and M' in which generators through P'' and P' intersect this generator. The first directrix of the line reflection is the line joining the fourth harmonic of M, M' and D_2 with D_2'. The second directrix is its conjugate polar and passes through D_2. The resolution of this displacement into the product of two line reflections is thus effected in ∞^2 ways.

(*c*) *One generator of each system remains fixed.*

This displacement has one fixed point A, the intersection of the fixed generators. It depends upon twelve parameters and is completely determined by the quadric S, the fixed point A and a pair of corresponding points P, P' on S. Through P and P' draw generators which will intersect one of the fixed generators in L and L'. Then upon this generator is established a projectivity such that $A^2 L \pi A^2 L'$. This projectivity may be resolved in ∞^1 ways into the product of two involutions with double points A, D_1 and A, D_2. The directrices of the line reflections are now determined as in case (*b*).

The projectivity may also be established upon the other fixed generator. The resolution of this displacement into the product of two line reflections may thus be effected in ∞^2 ways.

(*d*) *Two generators of one system and all of the second system remain fixed.*

All the points of the two fixed generators of the first system are invariant under this displacement which depends upon nine parameters. The displacement is completely determined by the quadric S, the fixed generators, and a pair of corresponding points which will lie upon a generator of S belonging to the second system. Then upon this generator is established a projectivity in which the intersections of the two fixed generators of the first system are double points. This projectivity may be resolved in ∞^1 ways into the product of two involutions with double points D_1, D_1' and D_2, D_2'. The directrices of the first line reflection are any pair of conjugate polars through D_1 and D_1'. They will be the opposite edges of a tangential tetrahedron formed by the generators through D_1 and D_1' and another generator of the second system. The directrices of the second line reflection are then uniquely determined. They are the conjugate polars which are opposite edges of a tangential tetrahedron formed by the generators through D_2 and D'_2 and the same generator of the second system. The resolution of this displacement into the product of two line reflections is thus effected in ∞^2 ways.

(*e*) *One generator of one system and all of the second system remain fixed.*

The points of the fixed generator of the first system remain invariant under this displacement which depends upon eight parameters. The displacement is completely determined by the quadric S, the fixed generator, and a pair of corresponding points which will lie upon a generator belonging to the second system. Then upon this generator is established a projectivity whose double points fall together in the intersection A with the fixed generator of the first system. This projectivity may be resolved in ∞^1 ways into the product of two involutions with double points D_1, A and D_2, A. The directrices of the line reflections are now determined as in case (*d*). The resolution of this displacement into the product of two line reflections may thus be effected in ∞^2 ways.

COMPOSITION OF TWO NON-EUCLIDEAN DISPLACEMENTS.

The resolution of non-euclidean displacements into the product of two non-euclidean line reflections affords a simple method of compounding two such displacements C_1 and C_2. For this it is necessary and sufficient that C_1 and C_2 be resolved so that a line reflection (nn') shall be common to both displacements. We may resolve two displacements C_1 and C_2 of type (*a*) into the product of two line reflections such that the unique transversals nn' of the axes of C_1 and C_2 are the directrices of a line reflection common to both displacements.* Then if $C_1 = (ll')(nn')$ and $C_2 = (nn')(mm')$, $C_1 C_2 = (ll')(nn')(nn')(mm') = (ll')(mm')$. When the transversals n and n' fall together the composition is evidently impossible. If C_1 belongs to type (*a*) and C_2 to type (*b*), the directrices (nn') of the line reflection common to both displacements are determined as follows: Through the fixed points of C_1 draw the generators which intersect the single fixed generator AB in L and L'. Find the points on this generator which divide harmonically L, L' and A, B. n and n' are the lines passing through these points and intersecting the axes of C_1. They are easily seen to be conjugate polars and are uniquely determined.

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* The unique transversals of two pairs of conjugate polars are conjugate polars, for if a line cut two pairs of conjugate polars the same is true of its conjugate polar.

The determination of the common line reflection for all the different cases presents no difficulties.

If in a non-euclidean displacement $C_1 = (ll')(mm')$, l intersects m, l' will intersect m'. C_1 has then an axis of fixed points which is the line joining the intersections of l with m and l' with m'. Conversely, every displacement having a fixed axis R may be resolved in ∞^2 ways into the product of two line reflections whose directrices intersect in pairs on the fixed axis. This follows at once from the fundamental theorem, if one notices that the corresponding points in the projectivity established upon R fall together as do the two sets of double points of the two involutions of which this projectivity is the product. Hence the directrices of the two line reflections intersect in pairs on R. Such a displacement will be called a *non-euclidean rotation* about the fixed axis. One sees readily that corresponding points lie in a plane with R', the conjugate polar of R.

RESOLUTIONS OF NON-EUCLIDEAN DISPLACEMENTS.

By means of a non-euclidean rotation and our fundamental theorem we readily obtain the following resolutions of non-euclidean displacements.

A straight line L being given every non-euclidean displacement may be resolved into the product of a non-euclidean rotation about L followed by a second non-euclidean rotation.

To prove this, resolve the displacement into the product of two line reflections $(ll')(mm')$ such that l shall be perpendicular to L. Let (nn') be a line reflection such that n is perpendicular to L at its point of intersection with l and let n also intersect m. Then $(ll')(nn')(nn')(mm') = (ll')(mm')$, and $(ll')(nn')$ is a rotation about L while $(nn')(mm')$ is a rotation about an axis R_1 which joins the intersections of n with m and n' with m'.

Let now (nn') be a line reflection such that n is perpendicular to L at its point of intersection with l and let n also intersect m'. Then $(ll')(nn')(nn')(mm') = (ll')(mm')$, and $(ll')(nn')$ is a rotation about L while $(nn')(mm')$ is a rotation about an axis R_2 which joins the intersections of n with m' and n' with m.

In these two resolutions the axis L is perfectly arbitrary and as far as the points in L are concerned the transformation is that of a non-euclidean rotation about R_1 or R_2. Hence

Every displacement of a straight line in non-euclidean space may be effected by a non-euclidean rotation about R_1 or about R_2.

NON-EUCLIDEAN SYMMETRY TRANSFORMATIONS.

To obtain corresponding theorems concerning the displacement of plane figures in non-euclidean space it is necessary to study non-euclidean symmetry transformations and in particular that known as a point plane reflection or a non-euclidean plane reflection. By this transformation a point P is transformed into a point P' such that the line PP' is divided harmonically by a point and its polar plane.*

The product of two non-euclidean plane reflections is a non-euclidean displacement having a fixed axis, the intersection of the two planes, and is therefore a non-euclidean rotation. The product of two plane reflections on perpendicular planes S_1 and S_2 passing through a line l is a non-euclidean line reflection having for directrices l and its conjugate l'. For if P is any point in space and P' the point obtained by reflection on S_1 and S_2, one may readily show that PP' is divided harmonically by l and l'. Conversely, every line reflection may be resolved into the product of two plane reflections on perpendicular planes S_1 and S_2. For if P and P' are corresponding points of the line reflection (ll'), pass a plane S_1 through l and P which intersects l' in M. S_2 is now chosen as the polar plane of M and P is transformed into P' by $(S_1)(S_2)$. Two plane reflections on perpendicular planes are commutative, for $(S_1)(S_2)(S_1)(S_2) = (ll')(ll') = 1$ and hence $(S_1)(S_2) = (S_2)(S_1)$.

Since a non-euclidean symmetry transformation interchanges the two systems of generators of the quadric, it is evident that *every non-euclidean displacement may be compounded of a non-euclidean plane reflection on an arbitrary plane followed by a non-euclidean symmetry transformation.*

Our fundamental theorem now enables us to obtain a form for symmetry transformations such that some of their important properties are at once evident.

Consider a plane reflection (S) followed by any non-euclidean symmetry transformation (Σ). The result is a displacement which may be compounded of two line reflections $(ll')(mm')$ such that l shall lie in the plane S. Replace (ll') by a plane reflection in S followed by a plane reflection in the perpendicular plane S_1 through l. Then $(S)(\Sigma) = (ll')(mm') = (S)(S_1)(mm')$.

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* It is readily seen that this transformation interchanges the two systems of generators of the quadric surface, for the tangent planes through the pole contain a generator of each system which are interchanged by the transformation.

Whence $(\Sigma) = (S_1)(mm')$ and $S_1(mm')$ is any symmetry transformation. Hence

Every non-euclidean symmetry transformation may be effected by a non-euclidean plane reflection followed by a non-euclidean line reflection.

Replace now (mm') by two plane reflections on perpendicular planes S_2 and S_3 through m such that S_1 and S_3 are perpendicular. Then $S_1(mm') = (S_1)(S_2)(S_3) = (S_3)(S_1)(S_2) = (R)(S_3) = (S_3)(R)$ where $(S_1)(S_2)$ is a rotation about an axis R perpendicular to S_3. Since (R) and (S_3) are commutative it follows that

Every non-euclidean symmetry transformation may be effected by combining in either order a non-euclidean plane reflection followed by a non-euclidean rotation about an axis perpendicular to the plane of the reflection.

The line reflection (mm') may also be replaced by two plane reflections on perpendicular planes S_4 and S_5 through m' such that S_1 and S_5 are perpendicular. Then, since planes through conjugate polars are perpendicular, the planes S_2, S_3, S_4, S_5 are mutually perpendicular and are the four planes of a polar tetrahedron. Since the plane S_1 passes through the pole of S_3 and S_5 the intersection of S_1 and S_2 is the same as that of S_1 and S_4. Then $S_1(mm') = (S_1)(S_4)(S_5) = (R)(S_5)$. Hence

Every non-euclidean symmetry transformation may be resolved into the product of a non-euclidean plane reflection on one of two perpendicular planes followed by a non-euclidean rotation about an axis perpendicular to these planes.

Every symmetry transformation $(R)(S_3) = (R)(S_5)$ has therefore two invariant perpendicular planes S_3 and S_5 and two invariant points on R, namely the poles of S_3 and S_5.

In the resolution of a non-euclidean displacement into the product of a plane reflection followed by a symmetry transformation, the plane of reflection is perfectly arbitrary, and as far as the points in this plane are concerned the transformation is that of a symmetry transformation. Hence

Every displacement of a plane figure in non-euclidean space may be effected by a non-euclidean plane reflection followed by a non-euclidean rotation about an axis perpendicular to the plane of the reflection.

METRIC PROPERTIES.

Let us now introduce non-euclidean distance and angle. We shall thus see that the correspondence between the transformations discussed here and those of euclidean space is complete. When the quadric is chosen for the absolute, the angle between two intersecting lines P and Q is given by

$$C \log (PQ,\ T_1T_2)$$

where C is an arbitrary constant and T_1 and T_2 are the tangents drawn from the point of intersection to the conic which is cut from the quadric by the plane of P and Q. The distance between two points P and P' is given by

$$C' \log (PP',\ I_1I_2)$$

where I_1 and I_2 are the points of intersection of PP' with the quadric.

If H_1 and H_2 are the foci of the involution defined by two pairs of corresponding points PP' and I_1I_2 of a non-euclidean transformation then $PI_1\,H_1H_2 \,\pi\, P'I_2\,H_1H_2$ and hence

$$(PH_1,\ I_1I_2) = (P'H_1\ I_2I_1) = (H_1P',\ I_1I_2)$$

and

$$(PH_2,\ I_1\,I_2) = (P'H_2,\ I_2I_1) = (H_2P',\ I_1I_2)$$

That is, the foci H_1 and H_2 are the middle points of the line PP'. Thus by a non-euclidean line reflection or plane reflection a point P is transformed into a point P' such that P and P' are equidistant from two fixed points. The fixed points in the former case are the intersections of PP' with the directrices of the line reflection, in the latter case the pole and the intersection of PP' with the plane of the reflection.

It is readily seen that any line intersecting the axis at right angles is carried by a given non-euclidean rotation through a constant angle. For the line l is transformed by the rotation $R = (ll')(mm')$ into a line n also perpendicular to the axis R. Then if T_1 and T_2 are the tangents drawn from the intersection of l and m to the conic which is cut from the quadric by the plane of l and m, the cross ratio $(ln,\ T_1\,T_2)$ remains constant. For the lines T_1 and T_2 will touch the quadric at the points of intersection of R', the conjugate of R, with the quadric. Then upon R' is established a projectivity in which the intersection of R' with l and n are corresponding points and its intersection with T_1 and T_2 double points. Hence the cross ratio $(ln,\ T_1\,T_2)$ is constant and therefore the angle between l and n is constant. This constant angle is equal to twice the angle between l and m for by (mm') l is carried into a line n and m remains fixed. The cross ratio $(lm,\ T_1\,T_2)$ becomes $(nm,\ T_2\,T_1)$; that is, the angle between m and n equals the angle between l and m. Hence the path curves of a rotation are circles in the non-euclidean sense in planes perpendicular to the axis of rotation.

RESULTING THEOREMS.

To obtain theorems concerning the displacement of a straight line in non-euclidean space, consider the rotations $R_1 = (ll')(mm')$ and $R_2 = (nn')(ss')$ by means of which this displacement may be effected. If P and P' are corresponding points of these rotations then the middle points of PP' will lie upon R_1' and R_2' the conjugate polars of R_1 and R_2. For draw l and n through P. Then P remains fixed by (ll') or (nn') and is transformed into P' by (mm') or (ss'). The middle points of P, P' are the intersections of m and m' (or s and s') with PP'. These two middle points coincide with the intersections of PP' with R_1' and R_2', since PP', R_1' and R_2' are in the same plane and m' (or s') intersects both PP' and R_1' (or PP' and R_2'). Hence

THEOREM I. *If a straight line be displaced in non-euclidean space the loci of the two middle points of chords joining congruent points are straight lines.*

Also since the polar planes of the points on a line pass through its conjugate polar we have

THEOREM II. *Planes perpendicular to these chords at their middle points intersect, each set in a straight line.*

The middle points will lie on the conjugate lines of the axes of the two rotations by which the displacement may be effected and the perpendicular planes will pass each set through those axes.

Similarly we obtain theorems concerning the displacement of a plane figure in non-euclidean space by considering the symmetry transformation $(S_1)(S_2)(S_3) = (S_1)(S_4)(S_5)$ by means of which the displacement may be effected. If P and P' are corresponding points in $(S_1)(S_2)(S_3)$ then one middle point of PP' will lie in S_3. For pass S_1 through P, then P remains fixed by (S_1) and goes into P' by $(S_2)(S_3)$. But $(S_2)(S_3)$ is a line reflection and hence one of the middle points lies in S_3. In the same way we see that the other middle point lies in S_5. Hence the theorems.

THEOREM III. *If a plane figure be displaced in non-euclidean space the loci of the middle points of chords joining congruent points are perpendicular planes.*

THEOREM IV. *Planes perpendicular to three chords at their middle points pass each set through a point.*

The middle points will lie in the invariant planes of the symmetry transformation by means of which the displacement may be effected and the perpendicular planes at these points will pass through the invariant points of the transformation.

For elliptic space, in which the imaginary surface defined by the real equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$ is taken for the absolute, the above four theorems hold without modification.

For hyperbolic space, in which the real surface with imaginary generators $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$ is taken for the absolute, only one set of middle points lies inside the absolute. These points are all real since the fundamental involutions are hyperbolic. The four theorems are so far modified for real figures that only one middle point and one locus appear in each theorem.

If the absolute degenerates into the imaginary circle at infinity one set of middle points recedes to infinity and the theorems become the four well known theorems of Chasles for euclidean space which are the same as those for hyperbolic space.

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